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Statistical Conjunction Analysis and Modeling of Low-Earth-Orbit Catalogued Objects

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The number of space debris objects in the low Earth orbits, especially in the sun synchronous orbits, is increasing, and the breakups in 2007 have added more risk to the satellites in this region. It is necessary to study the risk of conjunctions in these orbits to plan a future course of action regarding the control and mitigation of space debris objects. A statistical approach to obtain the number of conjunctions in a prescribed altitude bin considering various inclination bands is presented in this paper. The low Earth orbits, which are affected the most by the accumulation of space debris objects, are analyzed with special emphasis on sun synchronous orbits. The study is based on the catalogued objects from the two line element sets. It is observed that, after the major breakups in 2007, the number of conjunctions in the sun synchronous orbital region is highly significant. The second part of the study concentrates on the modeling aspects of spatial density and brings out a stochastic model based on a mixture of Laplace distributions. It is noted from the model that the fragmentation events in low Earth orbit during 2007 have redefined the pattern of spatial density distribution in the region below 1100 km.

Nomenclature

a = area parameter in the modified Laplace distribution
di = half-range of the inclination band under consideration, deg
$f(x)$ = probability density function of the Laplace distribution
H_a = apogee height, km
H_p = perigee height, km
i = inclination, deg
m = location parameter in the modified Laplace distribution
N = number of objects passing through a bin considering one orbit only
N_y = number of objects passing through a bin in a year considering the repeated orbits
p = weight parameter in the modified Laplace distribution
S = spatial density, number of objects in a year/km^3
s = scale parameter in the modified Laplace distribution
V = the volume of the spherical shell, km^3
X = independent variable in the estimation problem
Y = dependent variable in the estimation problem
Z = model fit value
θ = parameters to be estimated
θ_L = lower limit of the parameters considered for estimation
θ_U = upper limit of the parameters considered for estimation

I. Introduction

SINCE the first satellite launch in 1957 and the subsequent first satellite breakup in 1961, the number of space debris objects to date has been increasing at a considerable pace [1]. The intentional explosions, accidental breakups, increasing launch rates, and debris created due to other space activities pose a substantial threat to any meaningful space activities, including the human space missions [1]. The region most affected by the large-sized catalogued debris objects is the low Earth orbit (LEO) [1,2].

Conjunction is a point in the future when two objects in Earth orbit could possibly collide or come very close to each other. Conjunction analysis generally provides the prediction of risk assessment between the space objects for a possible prevention, mitigation, or maneuver [2,3]. Conjunctions of these objects with operational satellites is a primary concern for space-faring nations. An assessment of the number of conjunctions is of paramount importance in mission planning, selection of orbital bands, and possible mitigation plans [2,3].

This paper deals with statistical conjunction analysis. In this analysis, the risk assessment is based on the gross orbital properties of space objects as a sample space, rather than the individual (discrete) assessment of conjunction among the objects. In discrete analysis, the study identifies those secondary space objects that come closer to the primary object within a well-defined working or maneuver box [3,4]. This study provides an overall assessment of the conjunction threat in the different altitude bins considering different inclination bands, with attention given to the sun synchronous orbit (SSOs) considering the inclination bands of 95–105 and 98–99 deg.

Present-day conjunction analyses and close approach studies are handicapped by many factors, including the accuracy of two line elements (TLE), model inaccuracies incurred by assumptions on atmospheric density models, the orbital perturbing forces, and the time constraints required for thorough analysis. Hence, the three main factors affecting the soundness of the conjunction analysis are the precision of the data, the accuracy of the propagation models, and how early the predictions can be carried out [5]. Primary mission planning, say, the selection of operational orbital bands for a satellite, requires only a gross level conjunction analysis highlighting the risks involved in these orbits. This paper provides a procedure to carry out this first-hand analysis of conjunction risk assessment.

A discrete procedure is developed to estimate the number of debris objects (operational and nonoperational) passing through the altitude bins in LEO restricted in the altitude range of 200–2000 km. The study is extended to statistical conjunction analysis by inferring the number of conjunctions in warning boxes of different sizes. Finally, the paper brings out a model for the spatial density based on a mixture of Laplace distributions [6]. For ready reference, the Laplace distribution details are presented in the Appendix.

The analyses are carried out with two TLE sets obtained from the Space Track Web site‡, one during December 2006 and the other in August 2007, to assess the conjunctions in SSO and to model the

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‡Data available online at www.space-track.org [retrieved 9 October 2008].

changed pattern of spatial density due to recent fragmentations in LEO.

II. Recent Debris Breakups in Low Earth Orbit

The debris clouds created by recent breakups in LEO have added a large number of fragments to the LEO region. As reported in the literature and in *Orbital Debris Quarterly News*, the breakup involving the Fengyun-1C is the single major source of debris fragments in LEO during the past 50 years [7]. The breakup occurred when Fengyun-1C resided in an orbit of 845 by 865 km with an inclination of 98.65 deg and a revolution period of 102 min.

Another severe breakup was the explosion of a Russian Arabsat booster (Breeze-M upper stage, catalogue no. 28944) over Australia on 19 February 2007. Arabsat-4A was launched on a Proton rocket from Baikanour on 28 February 2006 and was a failed geosynchronous earth orbit launch due to upper-stage malfunctioning. Arabsat failed to reach the desired orbit and was deorbited at a later time. The Breeze-M upper stage was in an eccentric orbit with an inclination of 51.5 deg, which ranges from 49 to 14,975 km from Earth. U.S. Space Command tracked 1111 fragments, and it was estimated that space wreckage could remain in orbit for many years [7].

Overall, 10 fragmentation events were identified during 2007. The annual rate of increase in catalogued orbital debris objects was the largest in the last 50 years of space age. The resulting fragments from these breakups were spread out from 200 km to more than 4000 km in altitude, which significantly increased the number of conjunctions and collision probability. These debris fragments frequently transit the orbits of hundreds of operational spacecraft, including human spaceflight regimes, posing increased risks to current and future space operations.

The distribution of the number of objects in the 800–900 km altitude band for all inclinations from 0 to 180 deg (with August 2007 TLE data) is shown in Fig. 1. It may be noted that the maximum number of objects (nearly 45%) are in SSO.

III. Discrete Conjunction Analysis

The discrete conjunction calculation is done by first obtaining the TLEs for all catalogued objects. The trajectories are then propagated forward for a set period of time, usually a few days. The resulting ephemerides of the objects are compared with other space objects to identify the objects that come within a critical distance as determined by the warning boxes [2–4].

During the warning step, the prediction models usually sacrifice some accuracy for speed. The temporal distance between epoch and the point of conjunction can also hamper the precision of the close approach calculations. The objects predicted to enter the warning boxes are reassessed in closer intervals of time using more accurate algorithms and are checked against a maneuver box, which is smaller than the warning box. However, the maneuver box is many times larger than the primary object to provide a safety margin, as the locations of tracked space objects are not precise enough [3,5].

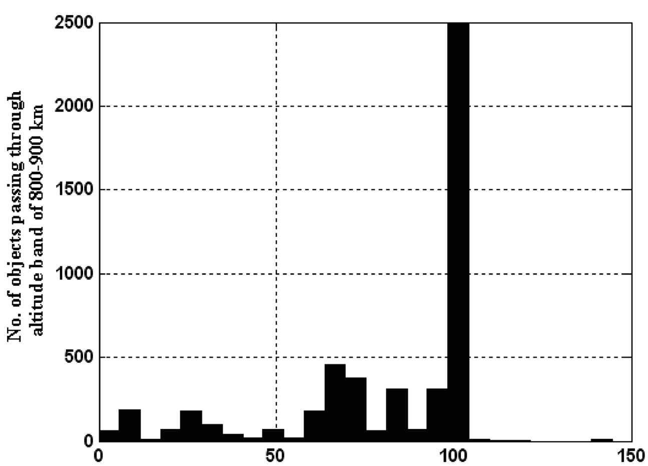

Fig. 1 Number of objects passing through the altitude band of 800–900 km.

Some spacecraft operators have already implemented collision avoidance schemes on some key space assets [8]. For example, NASA uses a warning box of approximately 25 km along the track of the orbit (either leading or trailing), 5 km across the track of the orbit, and 5 km radial from the Earth for the space shuttle. An estimated number of 10–30 objects per day that come within the warning box are reassessed using a more accurate algorithm to determine whether any of these come within a maneuver box of 5 km along track by 2 km in the radial direction. If an object does come within these parameters, the shuttle may initiate a maneuver to avoid collision [9].

IV. Present Procedure of Conjunction Assessment

To assess the number of objects passing through a particular altitude bin, [A km, B km], in a year, the following steps are followed:

1) Perigee filtering: The catalogued objects whose perigee was above B km were filtered out.

2) Apogee filtering: The catalogued objects whose apogee was below A km were filtered out.

3) Other objects, say, N, pass through the altitude bin [A km B km]. Assume that, in one orbit, the object crosses the bin twice on an average.

4) Number of objects in the bin in a year is defined as $N_y = 2N \times \text{norb}$, where norb is the number of times an object orbits the earth considering the repeated orbits in a year.

5) Spatial density is defined as $S = N_y/V$, where V is the volume of the spherical shell enclosed by the bin and S is the number of objects passing through the unit volume considering a time duration of one year.

6) Conjunctions in any warning box are obtained by considering the box's volume and altitude bin size on the longitudinal distance.

7) The number of conjunctions in a year for a target body (target orbit) is estimated assuming the residential period of the body in different altitude bins and integrating the spatial densities in the orbit passing through the altitude bins. For a near-circular orbit in an altitude bin, the number of conjunctions in a cubic kilometer box will be same as the spatial density.

Figures 2–5 give a sketch of four possibilities of debris objects' passage in the spherical shell between A km and B km in altitude. Figure 2 shows a situation in which the perigee of all the debris objects lies in the [A km, B km] band. Figure 3, shows a possibility in which the perigee of all those debris is less then A km and the apogee is greater than B km. Figure 4 represents all those debris objects whose perigee is less than A km and apogee is in the bin [A km, B km]. Figure 5 describes those circular or near-circular orbits that completely lie in the bin [A km, B km]. The perigee filtering, apogee filtering, and circular orbits within the [A km, B km] bin are also shown in these figures.

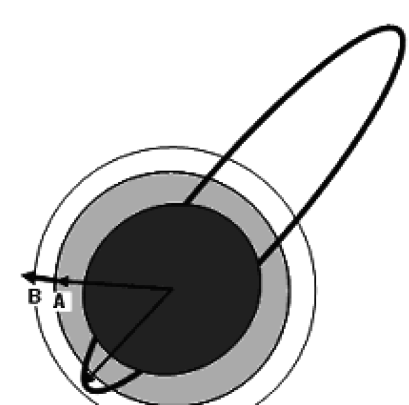

Fig. 2 Perigee is in bin [A km, B km].

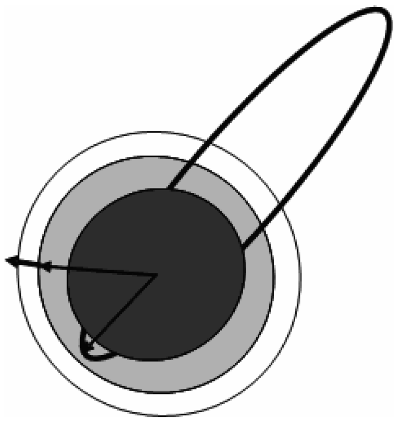

Fig. 3 Perigee is below A km and apogee above B km.

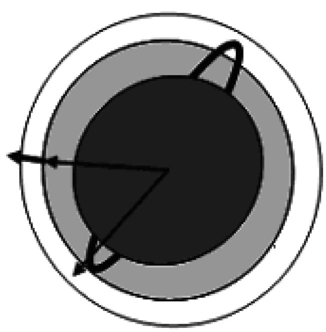

Fig. 4 Perigee is below A km and apogee is in bin [A km, B km].

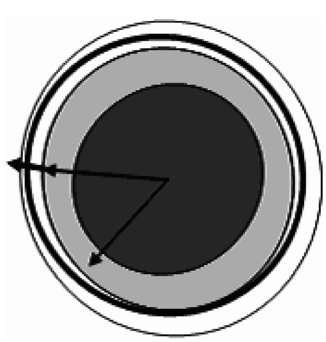

Fig. 5 Orbit is in bin [A km, B km].

Table 1 Number of conjunctions in a 5 × 1 × 1 km box per year at an inclination of 0–180 deg.

Altitude bin, km	Aug. 2007	Dec. 2006
800–820	0.3251	0.1848
820–840	0.3536	0.1963
840–860	0.4034	0.2032
860–880	0.3987	0.1961
880–900	0.3707	0.1959
900–920	0.3441	0.1995
920–940	0.3275	0.2010
940–960	0.3193	0.2130
960–980	0.3092	0.2219
980–1000	0.2887	0.2144

Table 2 Number of conjunctions in a 5 × 1 × 1 km box per year in the inclination region of 98–99 deg

Altitude bin, km	Aug. 2007	Dec. 2006
800–820	19.0944	7.0285
820–840	20.7969	7.1213
840–860	23.0639	5.1659
860–880	22.8377	3.4975
880–900	19.6935	2.9513
900–920	16.0741	2.5671
920–940	14.0314	2.2158
940–960	11.9639	1.9789
960–980	10.1558	1.5982
980–1000	8.5742	1.3294

Table 3 Number of conjunctions in a 5 × 1 × 1 km box per year at an inclination of 95–105 deg

Altitude bin, km	Aug. 2007	Dec. 2006
800–820	3.6337	1.0686
820–840	3.9659	1.0822
840–860	4.5271	0.8897
860–880	4.4330	0.7439
880–900	3.8885	0.7246
900–920	3.3506	0.6916
920–940	2.9707	0.6345
940–960	2.5697	0.6066
960–980	2.1702	0.5442
980–1000	1.8975	0.5097

V. Detailed Analysis

The aforementioned procedure is implemented to study the number of objects that pass through the band in a year and also to obtain the spatial density in the inclination bands of 0–180, 98–99, and 95–105 deg from the altitude range of 200–2000 km.

First, the 20 km bin was divided into four concentric spherical shells with a radial gap of 5 km between them. Then, the number of conjunctions in each of these subshells was found and integrated to calculate the number of conjunctions in the 20 km bin. The number of conjunctions for a warning box of size $5 \times 1 \times 1$ km in the altitude range of 800–1000 km for the inclination regions of study are given in Tables 1–3.

The maximum number of conjunctions in a warning box of size $5 \times 1 \times 1$ km (1 km radial, 5 km along the orbiter's track, and 1 km out of plane) were observed to be in the altitude range of 840–860 km (for different inclination regions of study with the TLEs of August 2007). A similar trend was observed with a warning box of size $1 \times 1 \times 1$ km. It may be clearly noted that the number of conjunctions was almost doubled after the Fengyun-1C breakup in the altitude range of 800–900 km.

It can be observed that the fragmentations in the first half of 2007 have significantly increased the number of objects in the altitude range of 800–1000 km; also, their spread can be observed in altitudes up to 1200 km. It may be seen that, for the inclination bands of 0–180 and 95–105 deg, the graphs clearly show a single peak, and the shift in peak position is observed from 800 (December 2006) to 820 km (August 2007).

A. In the Inclination Region of 0–180 Degrees

Figure 6 provides the number of objects passing through a 20 km altitude band. It also gives a comparison of the two TLE data sets under study. From this figure, it is clear that the number of objects have increased considerably in each of the altitude bands.

Figure 7 gives the number of objects passing through a 20 km altitude band in a year, taking into account the orbital period of each of the objects. It can be seen that the maximum number of objects are in the band of (820, 840) km.

Figure 8 provides the spatial density, the number of objects passing through the unit volume (1 km^3) in a year. It can be observed that the maximum spatial density is in the band of (820, 840) km.

B. In the Inclination Region of 98–99 Degrees

Figure 9 provides the number of objects passing through each 20 km band. It is noted that there is a slight shift in the location of the peak. Figure 10 presents the number of objects passing through each of the altitude bands in a year. Figure 11 gives the spatial density of the LEO objects in this inclination region.

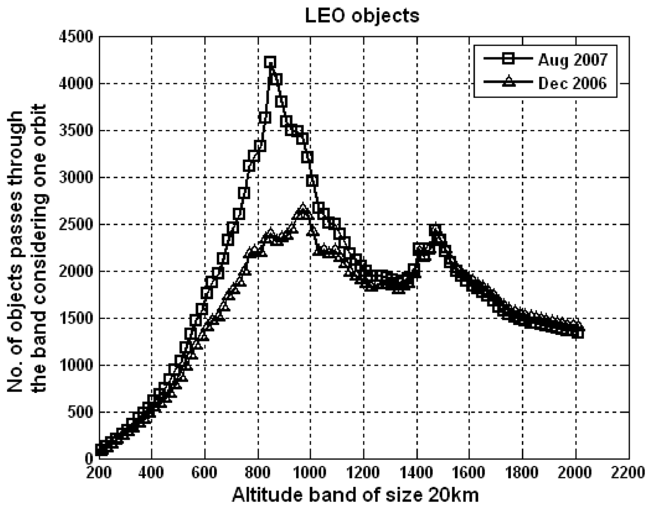

Fig. 6 Comparison of the number of objects at a 0–180 deg inclination through a 20 km altitude band considering one orbit for the August 2007 and December 2006 TLEs.

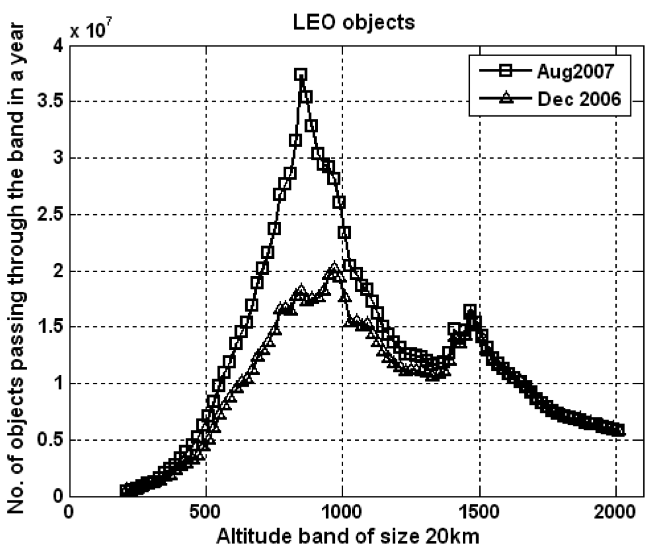

Fig. 7 Comparison of number of objects at a 0–180 deg inclination through a 20 km altitude band in a year for the August 2007 and December 2006 TLEs.

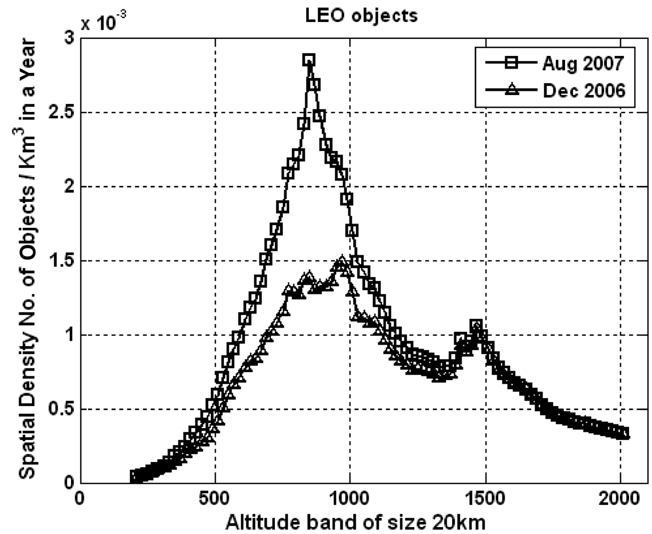

Fig. 8 Comparison of the spatial density (number of objects per cubic kilometer) at a 0–180 deg inclination through a 20 km altitude band in a year for the August 2007 and December 2006 TLEs.

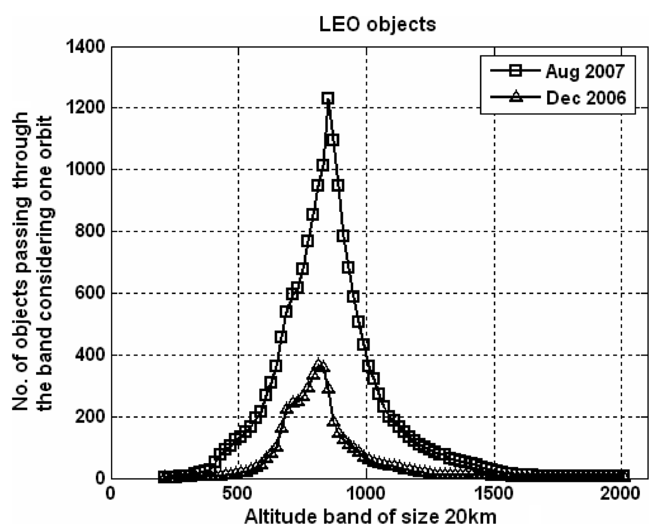

Fig. 9 Comparison of the number of objects at a 98–99 deg inclination through a 20 km altitude band considering one orbit for the August 2007 and December 2006 TLEs.

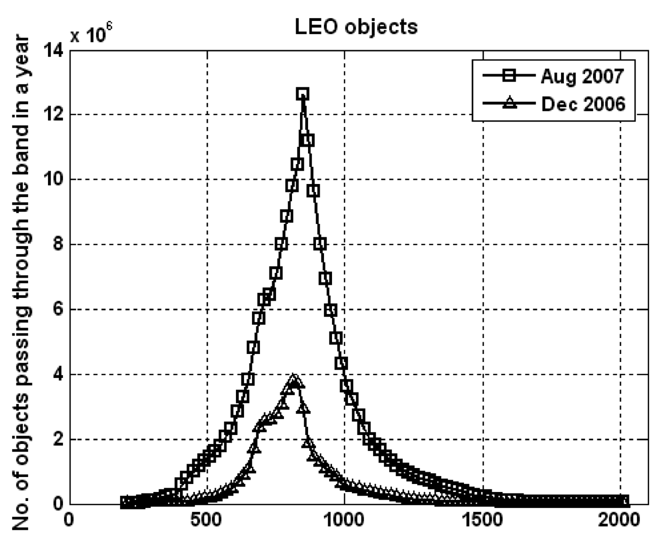

Fig. 10 Comparison of the number of objects at a 98–99 deg inclination through a 20 km altitude band in a year for the August 2007 and December 2006 TLEs.

C. In the Inclination Region of 95–105 Degrees

Figure 12 provides the number of objects passing through each of the altitude bands. Figure 13 describes the number of objects passing through each of the altitude bands in a year. Figure 14 provides the spatial density of the LEO objects in this inclination region. It can be observed that the shifts in these peaks are due to recent breakups that generated a large number of fragments concentrated at a certain altitude.

VI. Assessing the Risk Posed on a Target Body

Consider a target body with a perigee of H_p km and an apogee of H_a km through an inclination of i deg. Assume an inclination variation of di for this object. The steps to be followed for the estimation of the possible number of conjunctions are as follows:

1) Select the bin size of the altitude as the size of the warning box in the longitudinal direction.

2) Estimate the number of objects with an inclination of $(i - di, i + di)$ passing through each altitude bin in LEO in a year, as outlined earlier.

3) Identify the altitude bins between H_p and H_a that will be swept by the target body in an orbit.

4) Integrate the number of objects in each bin, taking into account the fraction of residential period in each bin.

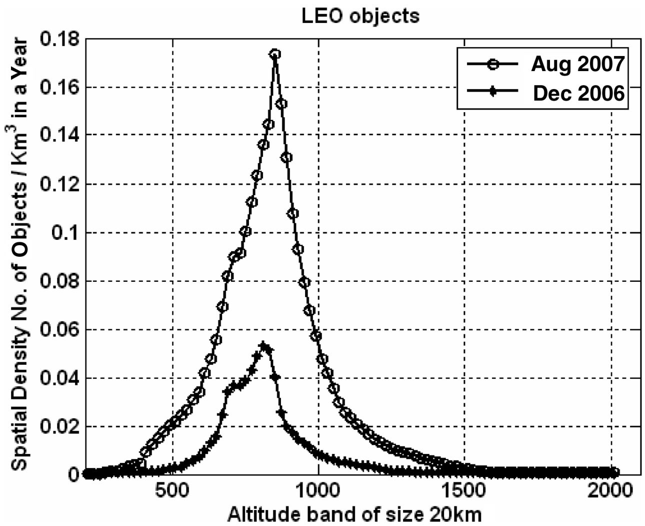

Fig. 11 Comparison of the spatial density (number of objects per cubic km) at a 98–99 deg inclination through a 20 km altitude band in a year for the August 2007 and December 2006 TLEs.

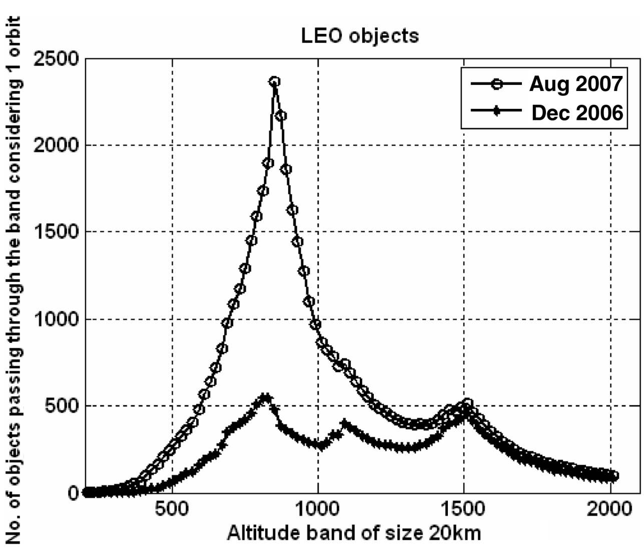

Fig. 12 Comparison of the number of objects at a 95–105 deg inclination through a 20 km altitude band considering one orbit for the August 2007 and December 2006 TLEs.

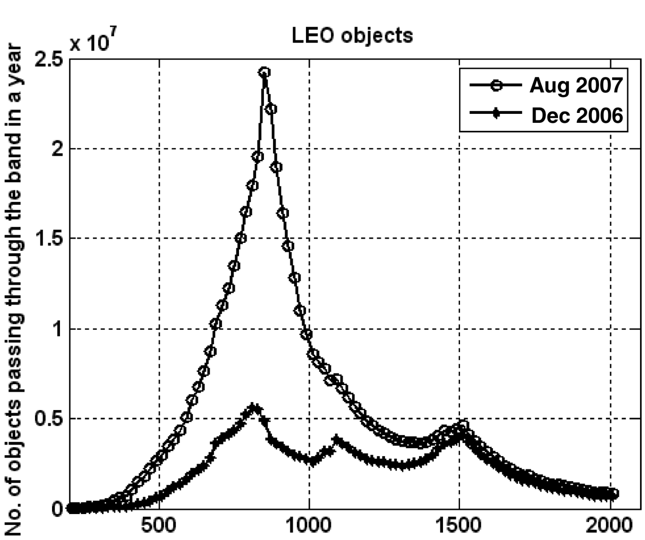

Fig. 13 Comparison of the number of objects at a 95–105 deg inclination through a 20 km altitude band in a year for the August 2007 and December 2006 TLEs.

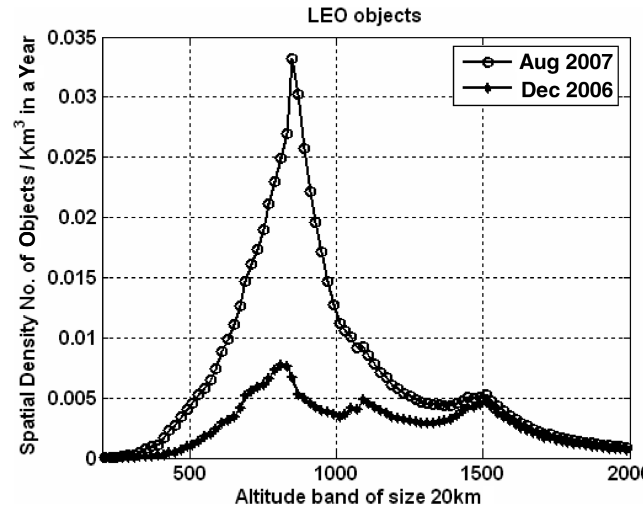

Fig. 14 Comparison of the spatial density (number of objects per cubic km) at a 95–105 deg inclination through a 20 km altitude band in a year for the August 2007 and December 2006 TLEs.

A. Case Studies

Consider a target body in SSO with an inclination in the 98.6–98.8 deg band and an orbit with a 500-km-high perigee and a 900-km-high apogee.

The number of conjunctions per year in a $5 \times 1 \times 1$ km box before and after the breakups in 2007 are 101.03 (with the December 2006 TLE data) and 261.31 (with the August 2007 TLE data), considering a 5 km altitude step from 500 to 900 km.

A CelesTrak conjunction analysis[§] estimates that about 11,000 conjunctions are possible within a week (7 days) among all catalogued objects with a $5 \times 1 \times 1$ km warning box.

As seen from the present analysis, one sun synchronous polar orbit object in (500, 900) km with a 98.7–98.9 deg inclination band has 261 conjunctions in a year, or about 5 conjunctions per week. Considering 5 conjunctions per week as a ballpark figure, one can calculate the total number of conjunctions per week as equal to the number of primary objects $\times$ 5 on average, which gives nearly 11,000 conjunctions, closely matching with the numbers quoted by the CelesTrak conjunction analysis.

B. A Specific Case Study

To assess the number of conjunctions per year, we chose the particular case study of Cartosat-1 (catalogue no. 28649), whose orbit at the time of injection had an inclination of 97.8 deg with a perigee of 619 km and an apogee of 623 km. The number of conjunctions in the inclination band of 97.75–97.85 deg (considering a ± 0.05 deg variation) is 6.53, considering a $5 \times 1 \times 1$ km warning box.

VII. Modeling of the Spatial Density

The spatial density is modeled using a modified Laplace distribution and mixtures of them [6,10–13]. The Laplace distribution was observed as the best-suited distribution for modeling the spatial density distribution [6,10]. The Laplace distribution model for the number of objects in the altitude bands were presented earlier [6,10] without considering the area parameters. In the present study, the parameters for the mixture of Laplace distributions are obtained by a random search method. The modeling of spatial density is useful for the fast prediction of the number of conjunctions in a prescribed altitude band.

A. Parameter Estimation Using a Random Search Technique

Here we present the method adopted to find the parameters of the proposed Laplace density model. Let (X_i, Y_i), $i = 1, \ldots, k$, where k is

[§]Data available online at www.celestrak.com [retrieved 9 October 2008].

the number of points in the given data, X is the independent vector, and Y is the dependent vector. Consider $Z_i = f(X_i, \theta_j, j = 1 \cdots m)$, where θ_j are the parameters for estimation as the model to be estimated for the given data [13]. The random search technique is adopted to search for those parameters that will minimize the sum of square errors, $\mathrm{SSE} = \sum_i (Y_i - Z_i)^2$, in a trial of a number of simulations (of the order of 10^5 or more). Initially, we begin the search for the required θ_j between the predefined lower limit θ_L and upper limit θ_U for each j. With this large a number of simulations, a sequence of parameter values, θ_j, for each j are generated that converge to a value that gives minimum square error.

B. Modified Laplace Distribution Functions

A modified version of the Laplace distribution introducing one more parameter, called *area parameter*, is considered for the modeling. The modified modeling function with this area parameter a is of the form

$$f(x) = \frac{a}{2s}\exp\left(\frac{-|x-m|}{s}\right)$$

For binary mixture, it is

$$f(x) = a\left[\frac{p}{2s_1}\exp\left(\frac{-|x-m_1|}{s_1}\right) + \frac{(1-p)}{2s_2}\exp\left(\frac{-|x-m_2|}{s_2}\right)\right]$$

And for the tertiary mixture, it is

$$f(x) = a\left[p_2\left[\frac{p_1}{2s_1}\exp\left(\frac{-|x-m_1|}{s_1}\right) + \frac{(1-p_1)}{2s_2}\exp\left(\frac{-|x-m_2|}{s_2}\right)\right] + \frac{(1-p_2)}{2s_3}\exp\left(\frac{-|x-m_3|}{s_3}\right)\right]$$

C. Results

The estimated parameters of the proposed models with the procedure described here are tabulated in Tables 4–6 respectively. The proposed models for the December 2006 data in the inclination bands of 0–180, 98–99, and 95–105 deg, respectively, are in the following form:

$$f(x) = 0.23653\exp\left(\frac{-|x-800|}{111.3}\right)$$

Table 4 Parameters of the model fit inclination of 98–99 deg

	Parameters		
TLE sets	Area a	$m1$	$s1$
Dec. 2006	13.163	800	111.3
Aug. 2007	52.497	850	151.86

Table 5 Parameters of the model fit inclination 0–180 deg

	Parameters					
TLE sets	Area a	$m1$	$s1$	$m2$	$s2$	$p1$
Dec. 2006	1.4121	900	291.08	1470	395.5	0.6
Aug. 2007	1.8706	850	200.61	1470	394.71	0.6

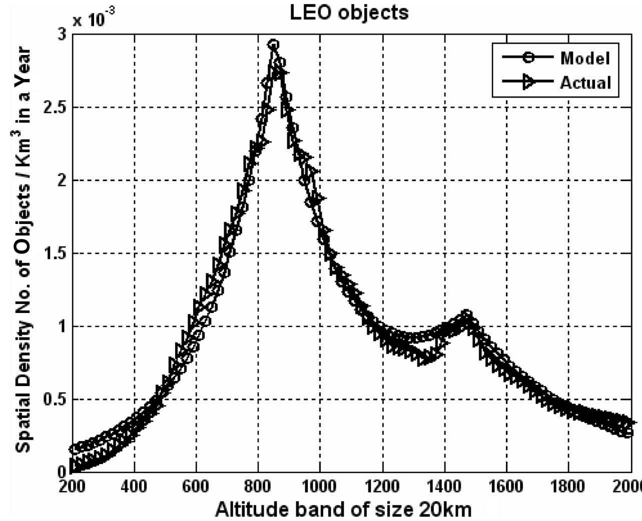

Fig. 15 Fit of binary mixture of the modified Laplace distribution for the spatial density to the August 2007 data at an inclination of 0–180 deg.

$$f(x) = 0.0015\exp\left(\frac{-|x-900|}{291.08}\right) + 7.1408\times10^{-4}\times\exp\left(\frac{-|x-1470|}{395.5}\right)$$

$$f(x) = 0.007108\exp\left(\frac{-|x-810|}{150.0}\right) + 0.003048\exp\left(\frac{-|x-1090|}{161.59}\right) + 0.007522\exp\left(\frac{-|x-1513.9|}{276.07}\right)$$

Similarly, the models for the inclination bands of 0–180, 98–99, and 95–105 deg can be derived from the TLEs of August 2007 using the parameter values mentioned in Tables 4–6. The model fits with the observed values corresponding to these TLEs for the three inclination bands are provided in the six graphs presented in Figs. 15–20.

Figure 15 shows the match between the observed and modeled spatial density for the August 2007 data at an inclination of 0–180 deg. Figure 16 presents the modified Laplace distribution model fitted for the spatial density in the inclination region of 0–180 deg considering the TLE data for December 2006. Figure 17 describes the model for spatial density to the August 2007 data in the inclination region of 98–99 deg; the single peak in this shows that the maximum spatial density is concentrated around 820 km. Figure 18 depicts the model fit for spatial density for the December 2006 data at an inclination of 98–99 deg. It may be observed that the height of the peak is less than half as compared with the peak from the August 06 data as shown in Fig. 17. Figure 19 represents the binary mixture of the Laplace model fit of spatial density for the August 2007 data in the inclination band of 95–105 deg. And Fig. 20 outlines the tertiary mixture model fit for the three peak spatial density distributions for the December 2006 data at an inclination of 95–105 deg. It may be noted that there is a substantial shift in the second peak after the breakups in 2007.

Table 6 Parameters of the model fit inclination of 95–105 deg

	Parameters								
TLE sets	Area a	$m1$	$s1$	$m2$	$s2$	$m3$	$s3$	$p1$	$p2$
Dec. 2006	5.3033	810	150.0	1090	161.59	1513.9	276.07	0.6999	0.5745
Aug. 2007	13.2	850	167.1	1470	254.80	–	–	0.80	–

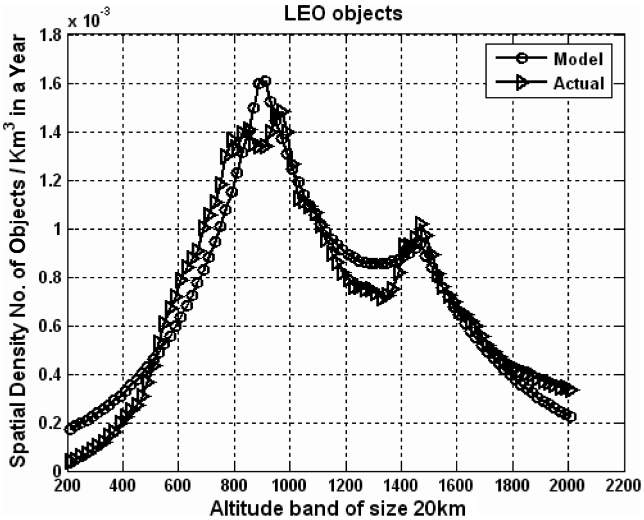

Fig. 16 Fit of binary mixture of the modified Laplace distribution for the spatial density to the December 2006 data at an inclination of 0–180 deg.

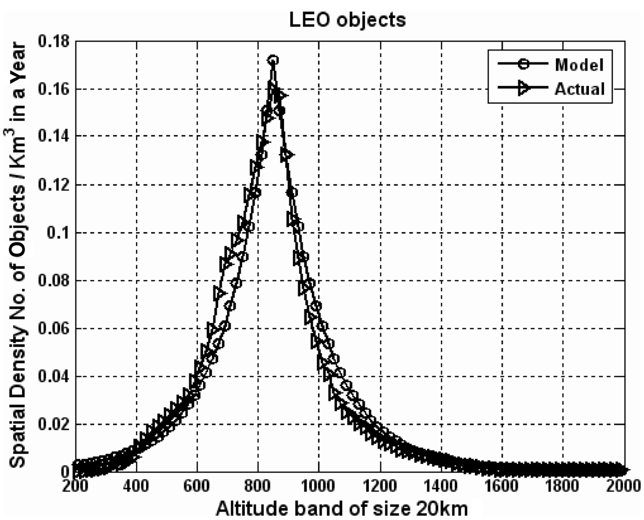

Fig. 17 Fit of modified Laplace distribution for the spatial density to August 2007 data in the inclination 98–99 deg.

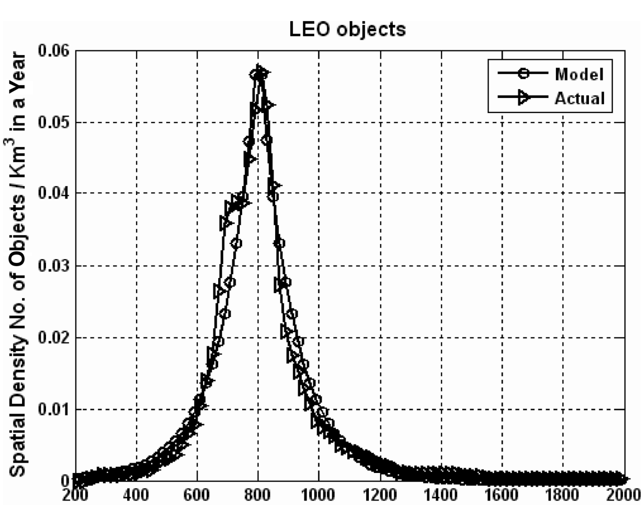

Fig. 18 Fit of modified mixture of Laplace distribution for the spatial density to the December 2006 data at an inclination of 98–99 deg.

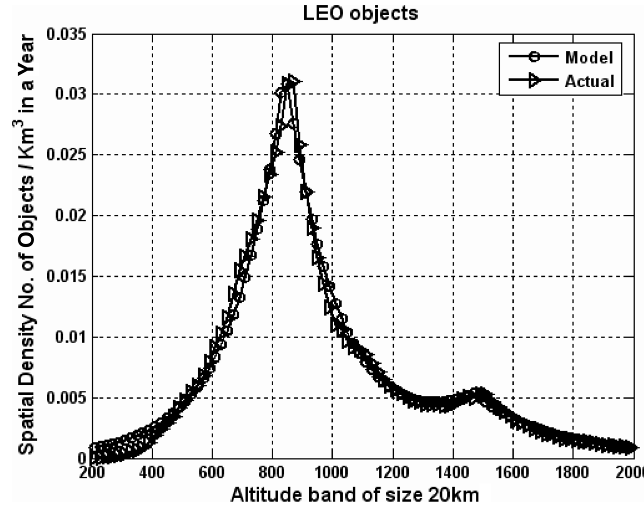

Fig. 19 Fit of binary mixture of the modified Laplace distribution for the spatial density to the August 2007 data at an inclination of 95–105 deg.

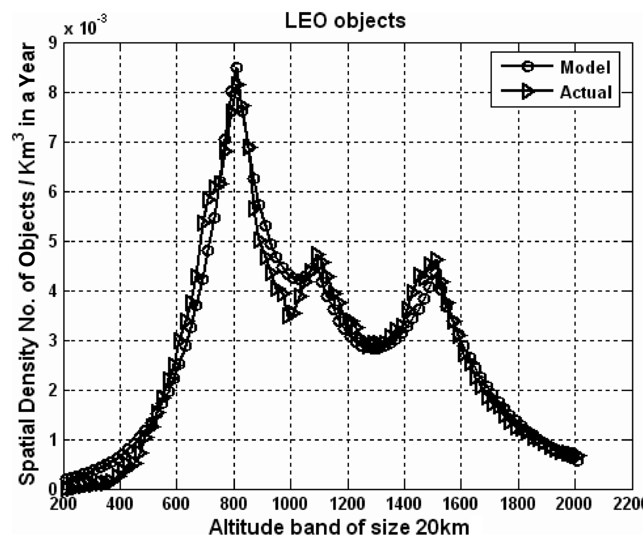

Fig. 20 Fit of tertiary mixture of the modified Laplace distribution for the spatial density to the December 2006 data at an inclination of 95–105 deg.

VIII. Conclusions

A procedure for assessing the number of conjunctions, in a statistical sense, in different orbital bands characterized by altitude and inclination is presented. The increasing threats of space debris conjunctions in sun synchronous orbits are highlighted. Further, an analysis of the impact of breakups on the number of conjunctions and a comparison of the scenarios after and before the major breakups in 2007 are also presented. A statistical model of the spatial density of the space debris objects in LEO based on the modified version of a mixture of Laplace distributions is explained. It is concluded that the major breakups that occurred in LEO during 2007 have redefined the pattern of spatial density in the LEO region.

Appendix: Laplace Distributions

The functional form of the probability density function (PDF) of the Laplace distribution as a function of two parameters, m and s, is

$$f(x) = \frac{1}{2s}\exp\left(\frac{-|x-m|}{s}\right)$$

Parameter m is called the *location parameter* and parameter s is called the *scale parameter*. Figure A1 shows a typical example with a location parameter value of 0 and a scale parameter value of 1.

The binary mixture of the Laplace distribution is

$$f(x) = \frac{p}{2s_1}\exp\left(\frac{-|x-m_1|}{s_1}\right) + \frac{(1-p)}{2s_2}\exp\left(\frac{-|x-m_2|}{s_2}\right)$$

And the tertiary mixture of the Laplace distribution [10,11] is

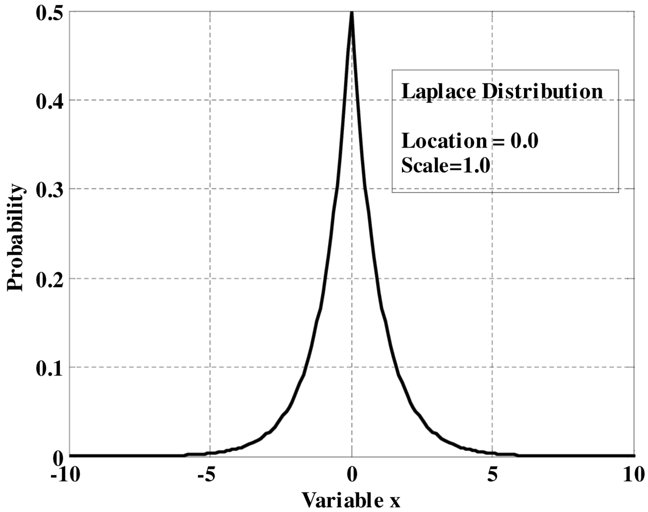

Fig. A1 Unitary Laplace distribution PDF.

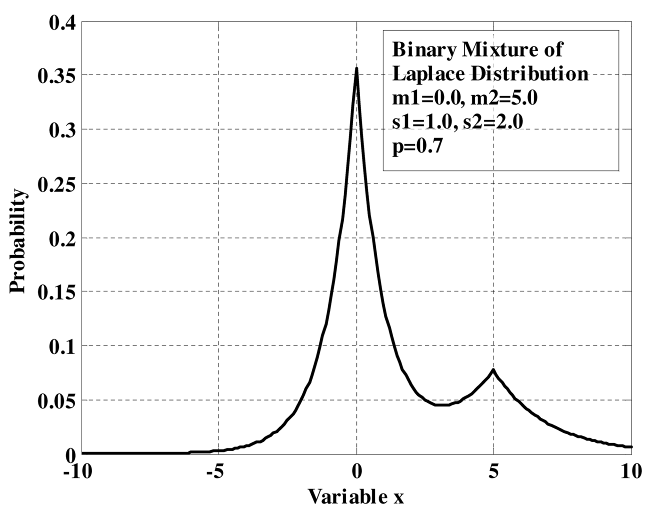

Fig. A2 Binary mixture of Laplace distribution PDF.

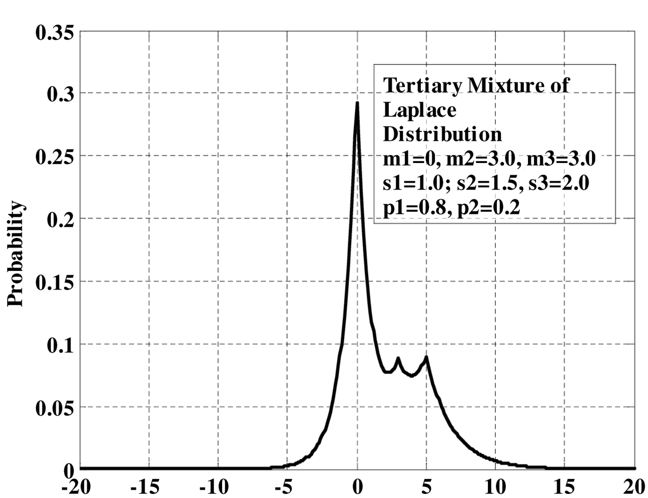

Fig. A3 Tertiary mixture of Laplace distribution PDF.

$$f(x) = p_2\left[\frac{p_1}{2s_1}\exp\left(\frac{-|x-m_1|}{s_1}\right) + \frac{(1-p_1)}{2s_2}\exp\left(\frac{-|x-m_2|}{s_2}\right)\right] + \frac{(1-p_2)}{2s_3}\exp\left(\frac{-|x-m_3|}{s_3}\right)$$

Figure A2 represents a binary mixture of the Laplace distribution with two scale and two position parameters and one weight parameter. Figure A3 shows the tertiary mixture of the Laplace distribution with eight parameters.¶

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¶Details about modeling with a mixture of Laplace distributions along with some applications are available at http://www.causascientia.org/math_stat/Dists/Compendium.pdf [retrieved 15 October 2008].